A Compressed Sensing Approach to Observing Distributed Radar Targets

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Compressed sensing, a method which relies on sparsity to reconstruct signals with relatively few measurements, has the potential to greatly improve observation of distributed radar targets. We extend the theoretical work of others by investigating the practical problems associated with this approach. First we derive a discrete linear radar model that is suitable for compressed sensing. The result is similar to the prior models, but we use the derivation to discuss what the model can and cannot tell us about distributed target signals. Then we provide details about how this problem can be solved in practice with large data sets. Lastly, we use this compressed sensing technique on existing data and discuss the results, showing that it works even when the data was collected with traditional approaches in mind. The potential benefits over those approaches include higher possible range resolution, complete filtering of noise without side-lobes or artifacts, and the ability to identify different Doppler shifts within the same range window of a single pulse.
1. Introduction

Compressed sensing [Donoho, 2006] is a new data acquisition and processing technique that leverages sparsity in the signal being measured in order to reduce the number of measurements needed to accurately reconstruct the signal. Many signals of interest are compressible and can be well-approximated by a relatively small amount of information when compared to their “raw” form. The current approach in many fields is to sample the data in its raw form and then compress and store it. Often it is only the “useful”, compressed information that was desired in the first place. Compressed sensing allows one to skip the inefficient raw sampling step and instead acquire an entire signal with an amount of information proportional to the signal’s compressed representation.

Because radar signals are quite recognizably sparse in range and frequency, with typically few targets of interest within range, radar is a natural target for compressed sensing. The role of sparsity in radar signal processing and how compressed sensing techniques relate to established processing methods is discussed by Potter et al. [2010] with an emphasis on synthetic aperture radar. The potential for compressed sensing to reduce radar hardware complexity and cost is noted by Baraniuk and Steeghs [2007] and Ender [2010], while Herman and Strohmer [2009] explores the use of compressed sensing for increased target detection resolution. It is this latter use that we are most interested in as a way to increase the resolution of meteor measurements made with high-power large-aperture (HPLA) radars.

When a meteoroid enters the Earth’s atmosphere, it collides with air molecules and heats up, causing ablation. This results in the formation of a plasma, called a meteor, which we
can measure with an HPLA radar due to electromagnetic scattering. The plasma surrounding the meteoroid is called a meteor head, while the plasma that is left behind is called a meteor trail. Unfortunately, the evolution of the plasma and the nature of the scattering from both the head and trail are not well understood and depend on the density of plasma and its orientation with respect to the background magnetic field. Often the meteor head, assumed to be small relative to the range resolution of the radar, is treated as a point scatterer, but this will not suffice for elucidating the more complicated aspects of meteor head echoes. Since in reality a plasma is a distributed collection of charged particles, we seek a measurement method that recognizes this and allows for high range resolution imaging.

Such a method requires a suitable radar model. A common approach, and one that we follow, is to discretize the target reflectivity in a joint time delay (or range) and Doppler frequency shift space. That is, we represent the received signal by a linear function of reflectivity coefficients, where each coefficient multiplies a time-delayed and Doppler-shifted version of the transmitted signal. Thus, the discrete signal is expressed in terms of a Gabor frame, a model which is efficient to compute and is compatible with the framework of compressed sensing.

Similar models for radar [Herman and Strohmer, 2009] and communication channels [Bajwa et al., 2008] have been used with compressed sensing previously. With the same goal of high resolution radar, Herman and Strohmer [2009] investigate the use of Alltop sequences as compressed sensing radar waveforms. For their model, they find that range and Doppler frequency resolution depend on the inverse of pulse waveform bandwidth and total sampling time, respectively. They also prove an upper bound on the target sparsity $s$.
for which solution is guaranteed with high probability and provide simulation results that indicate that the proven bound can be relaxed to $s \leq m/(2 \log m)$, where $m$ is the number of measurements. The development and results of Bajwa et al. [2008] proceed in much the same manner, except for the use of spread spectrum waveforms and the application to communication channels.

Both prior works provide a good foundation for using compressed sensing with radar from a theoretical perspective. What they lack are answers to more practical questions: How does the discrete model, essentially assuming point targets at very specific ranges and Doppler shifts, relate to a continuous radar model that allows distributed targets at arbitrary locations in the delay-Doppler space? How well does the technique work on real data which inevitably includes effects not present in the model? How can one implement the technique efficiently and with possibly large data sets? These are the questions that we set out to address in this paper.

Our development of a radar compressed sensing method begins with the derivation of a discrete linear radar model from a continuous one. From this, we find that solving using the discrete model gives an approximate lower bound on the total target reflectivity contained in a delay-Doppler window. The resolution of this window is determined by the pulse waveform bandwidth and the choice of Doppler discretization, the latter being limited only by the number of measurements through a compressed sensing solution condition.

We then describe how to implement our approach, solving for the target reflectivity using the large-scale optimization software TFOCS [Becker et al., 2010a]. Finally, we apply the method to ionospheric plasma data taken with the Poker Flat Incoherent Scatter Radar and
find that the solution agrees with that of a matched filter, validating the compressed sensing approach in a practical setting.

The benefits compressed sensing provides over traditional methods are many: (1) Range resolution is limited only by the transmission bandwidth, and is not directly constrained by sampling rate. (2) High range resolution and high sensitivity are decoupled and can both be achieved independently. (3) Signals are automatically separated from noise, without the chance of sidelobes or filter artifacts being misidentified as targets. (4) Identification of multiple Doppler shifts within the same range window of a single pulse is possible, which can resolve speeds within a distributed target. The cost of all of these advances is just processing complexity and time. No hardware upgrades are required to take advantage of compressed sensing, and it is even possible in some cases to reprocess existing data and gain new insights as our examples show.

2. Compressed Sensing Overview

Under the standard framework for compressed sensing, we seek to determine a vector signal \( f \in \mathbb{R}^n \) using \( m \) linear measurements with \( m \ll n \). Letting \( y \in \mathbb{R}^m \) denote the measurement vector, we can in general write its entries as an inner product \( y_k = \langle y, \phi_k \rangle \) for \( k = 1, \ldots, m \) for some \( \phi_k \in \mathbb{R}^n \). In matrix notation, we wish to solve for \( f \) in \( y = \Phi f \), where \( \Phi \) is the \( m \times n \) matrix with columns given by \( \phi_k \). Without further assumptions, this problem is ill-posed since ordinarily we would require \( m > n \) measurements to reconstruct \( f \). A simple typical case would be where \( \Phi \) is the identity matrix and the measurements \( y_k \)
are just the individual entries of the signal $f$. The surprising result of compressed sensing is that, under achievable conditions, the underdetermined problem with $m \ll n$ is solvable.

The first condition is compressibility or sparsity of the signal, which is a requirement that the signal be well-represented by a relatively small number of coefficients corresponding to elements in some dictionary. Given an orthonormal basis $\Psi = \{\psi_k \in \mathbb{R}^n : k = 1, \ldots, n\}$, we can write $f$ as $f = \sum_{k=1}^{n} x_k \psi_k$, where the coefficients $x_k = \langle f, \psi_k \rangle$ are given by the inner product between the signal and each of the basis vectors. We call $f$ compressible if there is some orthonormal basis such that $f \approx f^s = \Psi x^s$, where $\Psi$ is the matrix whose columns are the basis vectors $\psi_k$, $x^s$ is the vector of coefficients $x$ with all but the largest $s$ entries set to zero, and $s \ll n$. As the success of lossy compression schemes demonstrate, many signals of interest satisfy this condition.

The second condition is called incoherent sampling and pertains to the measurements of the signal. We restrict our attention to measurements given by the orthonormal basis $\Phi = \{\phi_k \in \mathbb{R}^n : k = 1, \ldots, n\}$. Given an orthonormal basis $\Psi = \{\psi_k \in \mathbb{R}^n : k = 1, \ldots, n\}$, define the coherence between $\Phi$ and $\Psi$ as

$$\mu(\Phi, \Psi) = n \max_{1 \leq j, k \leq n} \langle \phi_j, \psi_k \rangle^2. \quad (1)$$

This definition for the coherence comes from Candès and Romberg [2007], but $\mu$ can also be defined probabilistically or for linear measurements that are not given by an orthonormal basis (see Candès and Plan [2010]). It can be shown that $\mu$ ranges between 1 and $n$.

The incoherent sampling requirement is met when the coherence of the measurement set, $\Phi$, and the basis with which $f$ is compressible, $\Psi$, is close to 1. Conceptually, this condi-
tion ensures that the measurements are global in a sense, that each measurement contains information about almost every coefficient of the signal in the sparsity basis.

If the signal is compressible and incoherent sampling is performed, then essentially we know that each measurement contains a contribution from each of the $s$ coefficients. Intuitively, we might then expect to be able to reconstruct the signal from $s$ measurements. This idea is made concrete with the incoherent sampling theorem presented below in the case of reconstruction with the Dantzig selector. No matter the specific setting, the principle of this theorem remains the same: if the signal is compressible with $s$ coefficients and the sampling is incoherent, we can reconstruct the signal to within noise and approximation errors with a small constant times $s \log n$ measurements by solving a convex optimization problem.

Of the compressed sensing methods that account for noise and only approximately sparse signals, the Dantzig selector [Candès and Tao, 2007] is of particular interest to us. Let $A$ represent the measurement matrix whose $m$ rows are randomly sampled from a population of measurement vectors with coherence $\mu$ (for instance, these can be randomly sampled from the rows of $\Phi \Psi$ as defined above). Also, let the measurements be corrupted by noise given by $z \sim N(0, \sigma^2 I)$, a zero-mean i.i.d. Gaussian vector with variance $\sigma^2$. Therefore, the measurements are given by $y = Ax + z$. The Dantzig Selector is the solution to the optimization problem

$$\hat{x} = \arg \min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{subject to} \quad \|A^* (Ax - y)\|_\infty \leq \lambda \sigma$$

where $\lambda$ is a constant that is selected so that the actual signal obeys $\|A^* (Ax - y)\|_\infty \leq \lambda \sigma$ with high probability. Thus, the Dantzig selector finds a solution that has minimum $\ell_1$
norm, promoting sparsity, and is highly probable given the measurement noise. Finally, we have the incoherent sampling theorem, which guarantees that the Dantzig Selector provides a solution which is within noise error of the exact solution.

**Incoherent Sampling Theorem for Dantzig Selector** [Candès and Plan, 2010]: Suppose that a signal $x \in \mathbb{R}^n$ is measured as described above, and let $\lambda = 10\sqrt{\log n}$. Let $\beta$ denote a chosen constant, $\mu$ the coherence of equation (1), and $\bar{s}$ a chosen expected upper bound on the sparsity of $x$. If for a positive constant $C_0$ given by Candès and Plan [2010] (the exact value is not important for our purposes), the number of measurements $m$ satisfies

$$ m \geq C_0 (1 + \beta) \mu \bar{s} \log n, $$

then the Dantzig selector obeys

$$ \| \hat{x} - x \|_2 \leq \min_{s \leq \bar{s}} C_1 (1 + \alpha^2) \left[ \frac{\| x - x^* \|_1}{\sqrt{s}} + \sigma \sqrt{\frac{s \log n}{m}} \right] $$

$$ \| \hat{x} - x \|_1 \leq \min_{s \leq \bar{s}} C_1 (1 + \alpha^2) \left[ \| x - x^* \|_1 + s\sigma \sqrt{\frac{\log n}{m}} \right] $$

with probability at least $1 - 6/n - 6e^{-\beta}$ where $\alpha = \sqrt{(1+\beta)s \log n \over m}$, $C_1$ is a constant given by Candès and Plan [2010], and $x^*$ denotes the vector $x$ with all but its largest $s$ entries set to zero. So we see that the error of the solution is bounded by the error of any sparse solution ($\| x - x^* \|_1$, which goes to zero if $x$ is sparse) and the standard deviation of the measurement noise ($\sigma$).

### 3. Discrete Linear Radar Model

In order to use the Dantzig selector to get a compressed sensing solution for radar signals, we first need a discrete linear model describing the radar that meets the sparsity and
incoherence requirements. Appropriate models are presented by Herman and Strohmer [2009] and Bajwa et al. [2008] that discretize the radar signal in a joint time delay and Doppler shift space. Although we will use a model that is almost identical to those, the derivation that follows is nevertheless instructive because of what it tells us about how distributed targets fit within the model.

We begin with the narrow-band radar equation for (the complex envelope of) the received signal from a point scatterer:

\[ y(t) = s(t - t_d)e^{2\pi if_d(t - t_d)}e^{-2\pi if_0 t_d}h, \] (6)

where \( s(t) \) is the transmitted baseband modulation signal, \( f_0 \) is the transmission frequency, \( t_d \) is the time delay, \( f_d \) is the Doppler frequency shift, and \( h \) is the reflectivity coefficient of the point target, which must be a positive real number. If the scatterer has a range given by \( r \) and a range rate given by \( v \), then the time delay is approximately \( t_d \approx \frac{2r}{c} \) and the Doppler frequency is approximately \( f_d \approx -\frac{2v}{c}f_0 \) with \( c \) denoting the speed of light. If we let the reflectivity coefficient be a continuous positive real function of the time delay and Doppler frequency, then we can write the signal received from point scatterers distributed across time-Doppler space as

\[ y(t) = \int_{0}^{T} \int_{-B/2}^{B/2} s(t - t_d)e^{2\pi if_d(t - t_d)}e^{-2\pi if_0 t_d}h(t_d, f_d)df_ddt_d. \] (7)

The limits of the integrals are given by practical considerations: a radar only listens for so long before sending another signal (the inter-pulse period \( T \)), and the antenna and receiver are limited by a frequency bandwidth denoted as \( B \). Under this representation of the radar signal, we can think of a general, distributed target as having an associated reflectivity
function $h_\alpha(t_d, f_d)$ which describes how it scatters the radar signal back to the radar. What
the radar measures is the sum of these reflectivity functions for all targets, $h(t_d, f_d) = \sum_\alpha h_\alpha(t_d, f_d)$.

The next step is to discretize this model. We assume that the received signal is sampled
at a uniform rate $\tau_s$, so that $y(t)$ is represented by a complex discrete sequence $y_q = y(q\tau_s)$
with $q = 1, \ldots, m$ where $m = T/\tau_s$. In addition, we restrict our attention to discrete phase-
modulated signals with $b$ bauds and a baud length of $\tau_b$, so that $s(t) = s_k$ for $(k - 1)\tau_b \leq
< k\tau_b$ for a complex sequence $s_k \in \mathbb{C}$ with $k = 1, \ldots, b$. For ease of notation, let us
also infinitely extend the sequence $s_k$ by letting its value for non-existent indices be zero,
$s_k = 0$ for $k \neq 1, \ldots, b$. If we assume that the sampling time is an integer multiple of
the baud length, then we have $\tau_s = r\tau_b$ where $r$ is the under-sampling ratio. Note that
if the reverse is true and over-sampling by an integer ratio is performed, we can simply
duplicate the modulation sequence by the over-sampling ratio and let $r = 1$. In either case,
we know that $T/\tau_b = rm$. Since we have specified a sampling time of $\tau_s$, we know that
the bandwidth $B$ can be no greater than $1/\tau_s$. For simplicity, we ignore the unlikely case
that the receiver bandwidth is limited by a factor other than the sampling time and take
$B = 1/\tau_s$. Finally, without any further natural discretizations to guide us, we discretize
the frequency integral into segments of frequency $\Delta f$. This tells us that the number of
frequency steps is $n = \frac{B}{\Delta f} = \frac{1}{\Delta f \tau_s}$ where we are free to choose a value for $n$ or $\Delta f$.

With all of these assumptions, we can break up the time delay and Doppler frequency
shift integrals into segments and sum over them. With the goal being to remove as many
terms from under the integrals as possible, we manipulate equation (7) as follows:

\[
y(q\tau_s) = \sum_{k=1}^{rm} \int_{(k-1)\tau_b}^{k\tau_b} \sum_{p=-\frac{n}{2}+1}^{\frac{n}{2}} \int_{(p-1)\Delta f}^{p\Delta f} s(q\tau_s - t_d) e^{2\pi i f_d(q\tau_s - t_d)} e^{-2\pi i f_d t_d} h(t_d, f_d) df_d dt_d
\]

\[
= \sum_{k,p} s_{rq-k+1} \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} e^{2\pi i f_d q\tau_s} e^{-2\pi i (f_0 + f_d) t_d} h(t_d, f_d) df_d dt_d
\]

\[
= \sum_{k,p} s_{rq-k+1} e^{2\pi i pq} \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} e^{2\pi i q\tau_s (f_d - p\Delta f)} e^{-2\pi i (f_0 + f_d) t_d} h(t_d, f_d) df_d dt_d,
\]

where we have brought the complex exponential term out of the integrals in an attempt
to make the terms inside the integrals approximately independent of \(q\). We would like to
write this as

\[
y_q = \sum_{k=1}^{rm} \sum_{p=-\frac{n}{2}+1}^{\frac{n}{2}} s_{rq-k+1} e^{2\pi i pq} h_{p,k}
\]

\[
= \sum_{k=1}^{rm} s_{rq-k+1} \left( \sum_{p=-\frac{n}{2}+1}^{\frac{n}{2}} e^{2\pi i pq} h_{p,k} \right)
\]

where \(h_{p,k}\) give the entries of an \(n \times rm\) reflectivity matrix \(H\). Since as a function of \(p\),
\(e^{2\pi i pq}\) is periodic with period \(n\), we can rewrite this as

\[
y_q = \sum_{k=1}^{rm} s_{rq-k+1} \left( \sum_{p=0}^{n-1} e^{2\pi i pq} h_{p,k} \right)
\]

The outer sum represents a convolution in the time delay index between the transmitted
signal and reflectivity matrix. The inner sum is almost the inverse discrete Fourier trans-
form of the columns of the reflectivity matrix, as it is only missing a \(1/n\) coefficient. For
equation (10) to work, we need

\[
h_{p,k} \approx \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} e^{2\pi i q\tau_s (f_d - p\Delta f)} e^{-2\pi i (f_0 + f_d) t_d} h(t_d, f_d) df_d dt_d
\]

for all \(q = 1, \ldots, m\). The approximation will be good if either the reflectivity function
is nonzero only very close to the specific frequencies given by \(p\Delta f\) for \(p = -n/2 + \ldots, n/2\).
1, \ldots, n/2, or if the first complex exponential term stays close to 1 over the limits of the Doppler frequency integral, $e^{2\pi iq\tau_s \Delta f} \approx 1$. The latter is true if we have $q\tau_s \Delta f \ll 1$ for all $q = 1, \ldots, m$ or, in terms of the number of frequency steps, $n \gg m$.

Equation (10) is the discrete linear radar model that we were seeking. Radar targets, even the distributed ones, are typically localized enough in the delay-Doppler space that we can expect few of $h_{p,k}$ to be nonzero. Thus, this model readily meets the compressed sensing requirement of sparsity, in this case with respect to the standard basis for the delay-Doppler space. One question that remains is whether these measurements qualify as incoherent so that the guarantees of compressed sensing can be invoked. Of course, the answer to this question depends on the modulation sequence $s_q$. In practice, we have used this model successfully with random binary sequences ($s_q \in \{1, -1\}$) and the Barker-13 code. Alltop sequences were proven to result in incoherent measurements with a similar model [Herman and Strohmer, 2009]. Numerous papers discuss the incoherence properties of convolution or Toeplitz sensing matrices formed from random sequences [Bajwa et al., 2007; Romberg, 2009; Tropp et al., 2006] or chirp sequences [Tropp et al., 2006], a structure seen in the convolution portion of our model, while Candès et al. [2011] discusses sensing with Gabor frames, a feature which arises from the DFT portion of our model. So although we have not proven incoherence for any class of measurements made by this model, we nevertheless expect that many classes of modulation sequences will admit a compressed sensing solution.

Leaving the compressed sensing concerns behind, there is still the question of how well this discrete model describes a distributed target. One simple observation to make from
equation (11) is the following:

\[
|h_{p,k}| \approx \left| \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} e^{2\pi i q \tau_s (f_d - p\Delta f)} e^{-2\pi i (f_0 + f_d) t_d} h(t_d, f_d) df_d dt_d \right|
\]

\[
\leq \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} e^{2\pi i q \tau_s (f_d - p\Delta f)} e^{-2\pi i (f_0 + f_d) t_d} h(t_d, f_d) df_d dt_d
\]

\[
= \int_{(k-1)\tau_b}^{k\tau_b} \int_{(p-1)\Delta f}^{p\Delta f} h(t_d, f_d) df_d dt_d. \tag{12}
\]

Therefore, the absolute value of \( h_{p,k} \) gives us an approximate lower bound on the total target reflectivity contained in the time delay and Doppler frequency shift window given by \([(k-1)\tau_b, k\tau_b] \times [(p-1)\Delta f, p\Delta f]\). While this may not be a very tight lower bound (in fact one can imagine a distributed target with high total reflectivity that results in \( h_{p,k} = 0 \) because of destructive cancellations caused by the complex exponentials), it at least allows us some way of relating the measured discrete signals to a distributed reflectivity profile.

Another thing equation (11) tells us is what the discrete model cannot do. The first complex exponential in equation (11) varies in phase by an amount of \( 2\pi q \tau_s \Delta f \) over the Doppler shift integral, while the second complex exponential varies in phase by an amount approximately equal to \( 2\pi f_0 t_b \) over the time delay integral, where the approximation comes from an assumption that \( f_d \ll f_0 \). One way for \(|h_{p,k}|\) to be a good estimate of the target’s total reflectivity within the delay-Doppler window is if both of these phase shifts are small.

We already noted that this is true for the first complex exponential if \( n \gg m \). For the second complex exponential, this small phase shift requirement becomes \( t_b \ll 1/f_0 \). Making \( n \gg m \) is achievable, but potentially puts a strain on the number of measurements required in order to ensure that compressed sensing arrives at the correct result. The requirement on the baud length, however, is impossible to achieve. Since \( 1/t_b \) is limited by the radar’s transmission bandwidth, the requirement demands that the transmission band-
width be much greater than the transmission frequency! Thus, the only realistic way that
$h_{p,k}$ is a good estimate of the target’s total reflectivity within the delay-Doppler window
is if the reflectivity function is highly localized, which is exactly the case for point tar-
ggets. Nevertheless, the model remains useful even for distributed targets because of the
lower bound of equation (12) and the prospects of using it in conjunction with compressed
sensing techniques.

4. Implementation

To solve for the reflectivity, we employ a variation on the Dantzig selector called the
Gauss-Dantzig selector. First, an estimate is made with the Dantzig selector

$$\hat{h}^d = \arg\min_{h \in \mathbb{R}^n} \|h\|_1 \quad \text{subject to} \quad \|A^* (Ah - y)\|_\infty \leq \lambda \sigma$$  \hspace{1cm} (13)

where $h$ represents a vectorized form of $H$ and $A$ gives the linear measurements according
to the radar model. Then the locations of the non-zero components are taken from this
initial estimate and used to solve the constrained least-squares problem

$$\hat{h}^g = \arg\min_{h \in \mathbb{R}^n} \|Ah - y\|_2 \quad \text{subject to} \quad h_k = 0 \quad \forall k \text{ such that } \hat{h}^d_k = 0$$  \hspace{1cm} (14)

where we have constrained the solution $\hat{h}^g$ to only allow non-zero entries in the same
locations as $\hat{h}^d$. The effect of this procedure is to maintain the compressed sensing per-
performance of the Dantzig selector, notably the sparse solution and its robustness to noise,
while achieving an unbiased estimate of the non-zero components which the Dantzig se-
lector alone (biased toward zero) does not achieve.

Because of the potentially large dimensions of these convex optimization problems, with
$h$ having thousands of elements or more, second-order solution methods are often not fea-
sible. Therefore, it is necessary to pursue first-order gradient-based methods of optimization. Along with these large dimensions comes a need for efficiently computing the linear measurements represented by $A$. Simply taking the model of equation (10) and converting it to matrix form as $y = Ah$ will not work very well; with a matrix of that size, most computers will run out of memory very quickly. Luckily, the linear operator in this case is highly structured, and we can take advantage of this in the computations if given the opportunity.

As one might imagine, these problems are not unique to our case, so general-use software packages for compressed sensing and other large-scale optimization problems are available. The one we have chosen to use is called TFOCS (Templates for First-Order Conic Solvers) [Becker et al., 2010a], and it was developed to solve problems of the smoothed conic form described by Becker et al. [2010b]. Its benefits are that it is easy to specify the optimization problem and it allows one to provide an efficient implementation of the linear operator $A$.

The way we achieve this efficient implementation is to break the operator of equation (10) into two steps: first applying the Doppler frequency shift, then applying the time delay and convolution with the transmitted signal. For the first operation, we are referring to the calculation

$$g_{q,k} = \sum_{p=0}^{n-1} e^{\frac{2\pi i pq}{n}} h_{p,k}. \quad (15)$$

This is just an inverse discrete Fourier transform applied to the columns of $H$ and multiplied by a factor of $n$, so it is readily implemented using the FFT (fast Fourier transform)
algorithm. For the second operation, we must implement the function

\[ y_q = \sum_{k=1}^{r_m} s_{r_q-k+1}g_{q,k}. \] (16)

Although it might be tempting to try to take advantage of the convolutional structure of this sum and once again use the FFT algorithm, this would actually involve performing \(m\) convolutions and discarding most of the resulting values because the \(g\) term depends on \(q\) as well as \(k\). Thus the straightforward approach of performing the sum directly is the correct approach. For both of these operations, one can take advantage of sparse data structures to minimize memory and computation even further. Though these observations are trivial, the difference in computation time between using this efficient implementation and using either a brute-force sum or a giant matrix is certainly not trivial.

One difficulty in using TFOCS is that it actually solves a smoothed version of the optimization problem, which is necessary because both the Dantzig selector and constrained least squares problems are not differentiable everywhere. Thus there is a need to select a value for the smoothing parameter \(\mu\) which weights the smoothing term of the minimization objective function. If \(\mu\) is too large, the optimization converges to an incorrect value; if \(\mu\) is too small, the optimization converges too slowly. Finding the sweet spot requires trial and error, although this may improve in future versions of TFOCS. In our experience with the examples to follow, letting

\[ \mu = \frac{\|A^*A\|^2}{\|A^*y\|_F^2} \] (17)

strikes the balance reasonably well in an automated fashion for each individual measurement vector \(y\).
In total, our compressed sensing implementation of distributed radar target measurement proceeds as follows. Data is collected using a discrete phase shift pulse waveform that results in sufficiently incoherent measurements, with baud length determining range resolution and sampling rate setting the limit for feasible reconstruction. Processing the data begins with choosing a Doppler shift resolution, with the only constraint being that enough measurements have been collected with respect to the chosen Doppler resolution to invoke the incoherent sampling theorem and guarantee solution accuracy. Knowing the dimensions of the problem, two linear functions are implemented to efficiently calculate the measurements from the reflectivity matrix according to our model. Then the locations of the non-zero entries of the reflectivity matrix are found by using TFOCS to solve the Dantzig selector with smoothing parameter $\mu$ given by equation (17). The final solution is reached by solving the sparsity-constrained least squares problem that completes the Gauss-Dantzig selector. The resulting reflectivity matrix tells us approximately how much signal was returned by each corresponding window in delay-Doppler space.

5. Examples

While some testing of this procedure was initially done using simulated measurements generated according to the discrete linear radar model, the only way to find out if the model and procedure work in practice is to test them on real data. Because data collected with compressed sensing specifically in mind is in the planning stages and unavailable at the time of writing, we turn to existing data to demonstrate the method.
Our two examples are taken from two hours of data obtained by the Poker Flat Incoherent Scatter Radar on July 28, 2010. The measurements were made using a Barker-13 code with a baud length of 10 microseconds and a sampling period of 5 microseconds. It should be stressed that these parameters are not ideal for compressed sensing, as they were chosen with matched filter processing in mind. Nevertheless, the Barker-13 waveform is a discrete phase shift code as required by our model, and it results in measurements that are sufficiently incoherent so that using compressed sensing is possible.

The first example is of a particularly strong meteor head echo. Depicted in Figure 1 is the SNR (signal to noise ratio) of the meteor head as a function of range and pulse time as given by the matched filter (Figure 1a) and compressed sensing (Figure 1c). Of course, the compressed sensing solution also yields the SNR as a function of Doppler frequency, and this is shown in Figure 1d. In order to arrive at the matched filter result, it is necessary to try multiple filters that have each been frequency shifted by a different amount in order to account for the Doppler shift of the returning signal. The single matched filter result is then taken to be the one that results in the highest SNR out of all of the shifted filters. This maximum SNR and the corresponding frequency shift are shown in Figure 1b for comparison with the compressed sensing result. Figure 2 shows a similar set of results for an unidentified event in the E-Region of the ionosphere. We chose this example for the sake of variety, to demonstrate the compressed sensing results on something which is definitely not a meteor head echo. The cause of this event, although surely interesting, is unimportant to this goal.
The first thing to note about these examples is that the matched filter and compressed sensing results agree, showing approximately the same SNR for the signal at the same locations in range and Doppler frequency shift. Although it is only two examples, at the very least this tells us that our approach is valid and can work on real data sets. Perhaps the second most striking takeaway from these results is the lack of noise in the compressed sensing solution, especially when compared to the range sidelobes most notably present in Figure 1a. This is a natural consequence of the compressed sensing approach, which searches for a sparse solution that falls within noise bounds. A similar effect could be achieved in the matched filter case by setting all SNRs below a certain noise threshold to zero, and this is often done in practice to separate signals from noise. The difference is that with compressed sensing, it is not possible to misidentify filtering artifacts (such as range sidelobes) as signal. The third important observation is that the compressed sensing solution associates multiple Doppler frequency shifts with each pulse, whereas the matched filter is limited to one. As in the case of the meteor head echo seen in Figure 1d, this can tell us that different portions of the plasma are moving at different speeds, resulting in a range of Doppler frequency shifts that we can resolve with compressed sensing. This is perhaps the biggest immediate benefit that can be had by employing our technique.

6. Conclusion

Compressed sensing provides an exciting new way to look at radar signals. The radar model that we derived provides insight into the relationship between the compressed sensing solution and a distributed target’s reflectivity in the form of an approximate lower
bound of the total reflectivity in a time delay and Doppler shift window. Our model is very similar to the previous models by Herman and Strohmer [2009] and Bajwa et al. [2008] for which compressed sensing has been explored theoretically, lending mathematical support to our procedure. The efficient implementation the we developed for this procedure allows its use on large data sets, which is an important step to analyzing real-world data. From our two examples of a meteor head echo and an unidentified E-region event using data from the Poker Flat Incoherent Scatter Radar, we know that the compressed sensing procedure works and can provide new insight compared to current techniques.

One of the many benefits of using compressed sensing for radar is the ability to discern multiple Doppler shifts not only within the same pulse but also within the same range window. For meteor studies with HPLA radars, this ability is vital to elucidating the complex processes present in the plasma. In addition, compressed sensing provides the opportunity for higher range resolution when compared to traditional techniques since the time delay resolution of its solution is not directly limited by the sampling period. The high range resolution can be achieved with high sensitivity since compressed sensing waveforms allow one to choose baud length and pulse length arbitrarily and independently. A natural consequence of compressed sensing’s search for a sparse solution is that the result identifies the location of signal only, filtering out all noise without sidelobes or other artifacts.

While these benefits are a nice step forward, there is still lots of room for improvement. Realizing these benefits on real-world data is an obvious first step and one that we are taking currently by collecting HPLA radar data with compressed sensing specifically in mind. For better sensing of distributed targets, we need a model that is more suited to
them and avoids the approximations made in our model. To this end, we will also be
investigating the use of wavelets as an appropriate basis for distributed target signals. Even
though there is much work to be done, the future of compressed sensing of distributed
radar targets looks promising indeed.

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Figure 1. Decoding of Strong Meteor Head Echo
Figure 2. Decoding of Unidentified E-Region Event